DAMTP/97-1
hep-th/9703196

Modified Weak Energy Condition for the Energy Momentum Tensor in Quantum Field Theory

José I. Latorre* and Hugh Osborn**

* Departament d'Estructura i Constituents de la Matèria,
Diagonal 647, 08028 Barcelona, Spain

** Department of Applied Mathematics and Theoretical Physics,
Silver Street, Cambridge, CB3 9EW, England

The weak energy condition is known to fail in general when applied to expectation values of the the energy momentum tensor in flat space quantum field theory. It is shown how the usual counter arguments against its validity are no longer applicable if the states $|\psi\rangle$ for which the expectation value is considered are restricted to a suitably defined subspace. A possible natural restriction on $|\psi\rangle$ is suggested and illustrated by two quantum mechanical examples based on a simple perturbed harmonic oscillator Hamiltonian. The proposed alternative quantum weak energy condition is applied to states formed by the action of scalar, vector and the energy momentum tensor operators on the vacuum. We assume conformal invariance in order to determine almost uniquely three-point functions involving the energy momentum tensor in terms of a few parameters. The positivity conditions lead to non trivial inequalities for these parameters. They are satisfied in free field theories, except in one case for dimensions close to two. Further restrictions on $|\psi\rangle$ are suggested which remove this problem. The inequalities which follow from considering the state formed by applying the energy momentum tensor to the vacuum are shown to imply that the coefficient of the topological term in the expectation value of the trace of the energy momentum tensor in an arbitrary curved space background is positive, in accord with calculations in free field theories.

PACS: 11.10-z; 11.25.Hf; 11.10Kk; 04.62+v

emails: latorre@ecm.ub.es and ho@damtp.cam.ac.uk

1. Introduction

In classical general relativity various positivity conditions on the energy momentum tensor for matter play an essential role in the proof of singularity, and other, theorems (see [1]). The simplest and perhaps most natural such inequality is the weak energy condition which asserts that $n^\mu n^\nu T_{\mu\nu}(x) \geq 0$ for n^μ any timelike vector at x. In a quantum theory the natural extension of such a condition is to require that it should be true for the expectation value of $n^\mu n^\nu T_{\mu\nu}(x)$ where $T_{\mu\nu}$ is the operator representing the energy momentum tensor which is defined in any quantum field theory. However it was soon realised [2] that such a condition must in general fail even on topologically trivial space-times (see the appendix in [3]), apart from the possibility of negative energy densities in the Casimir effect or in the neighbourhood of the event horizon of black holes [4].

A version of the argument [3] showing this follows simply in the context of elementary quantum mechanics by considering an hermitian operator T and a state $|0\rangle$ such that

$$\langle 0|T|0\rangle = 0\,, \qquad T|0\rangle \neq 0\,. \tag{1.1}$$

It is then evident that the positivity condition,

$$\langle \psi|T|\psi\rangle \geq 0\,, \tag{1.2}$$

cannot be true for all states $|\psi\rangle$. This is perhaps obvious by virtue of the standard variational principle for determining the lowest eigenvalue of T but a formal proof may be obtained by considering

$$|\psi\rangle = |0\rangle + \epsilon|\phi\rangle\,. \tag{1.3}$$

Then, assuming $\langle 0|T|\phi\rangle \neq 0$,

$$\langle \psi|T|\psi\rangle = 2\epsilon\, Re\langle 0|T|\phi\rangle + \epsilon^2\, \langle \phi|T|\phi\rangle < 0\,, \tag{1.4}$$

for some region of small $\epsilon > 0$ or $\epsilon < 0$.[1] However this counterexample to the general applicability of (1.2) fails if we impose the restriction[2]

$$\langle 0|T|\psi\rangle = 0\,. \tag{1.5}$$

[1] It is perhaps illuminating to consider the manifestly positive operator x^2 in simple quantum mechanics. If $\langle\psi_\lambda|x^2|\psi_\lambda\rangle = \lambda > 0$, for a normalised state $|\psi_\lambda\rangle$, then the above argument for $|0\rangle \to |\psi_\lambda\rangle$ and $T \to x^2 - \lambda$ shows that there exists a state $|\psi_{\lambda'}\rangle$ with $\lambda' < \lambda$ but there is no state in the Hilbert space giving $\lambda = 0$.

[2] It would be nice if $\langle\psi|T|\psi\rangle$ could be bounded below in terms of $\langle 0|T|\psi\rangle$, on the basis of (1.4) we considered $\langle\psi|T|\psi\rangle \geq -2|Re\langle 0|T|\psi\rangle|\,||\psi||$ but this can also be shown to be incompatible with general principles.

In order to say something in a semi-classical context it is clear that some restrictions on the applicability of classical energy conditions are necessary. Ford and coworkers [5] have suggested a non local condition involving an integral over a timelike or possibly null geodesic and which bounds the extent to which the expectation can be negative in terms of the width of the time averaging. The resulting inequalities have been verified for free field theories on flat space-time and may be compatible with extending the proof of some singularity theorems to semi-classical general relativity. However these inequalities are formulated entirely in a Minkowski space framework and it is not all clear how they might be applied to the Wick rotated quantum field theory on Euclidean space (any non perturbative definition of quantum field theory tends to be initially in a Euclidean framework).

Alternatively we here postulate a non standard quantum version of the weak energy condition which is motivated by considering how to make the above proof of the breakdown of the simple extension to quantum theory inapplicable. Thus we propose applying the condition only to a subspace of the full Hilbert space, thus

$$\langle\psi|n^\mu n^\nu T_{\mu\nu}(0)|\psi\rangle \geq 0\,, \qquad n^\mu = (1,\mathbf{n})\,, \quad \mathbf{n}^2 \leq 1\,, \tag{1.6}$$

for all states $|\psi\rangle$ satisfying

$$\langle 0|n^\mu n^\nu T_{\mu\nu}(0)|\psi\rangle = 0\,, \tag{1.7}$$

and for our purposes $|\psi\rangle$ is given by the action of finitely many local field operators acting on the vacuum state $|0\rangle$. By considering simple examples in appendix A we show that there is no inherent contradiction in making this assumption with the basic principles of quantum mechanics. In this paper we demonstrate that this quantum energy condition can lead to non trivial constraints by considering its application to a quantum field theory at a conformal invariant fixed point. The virtue of assuming conformal invariance is that the two-point functions of quasi-primary fields are then uniquely determined up to an overall constant and there are in general a finite set of linearly independent expressions for the three-point function (in some cases there is only one possible form, such as when two fields are scalars). This analysis also allows the consideration of free massless scalar and fermion field theories in arbitrary dimension d and free vector fields for $d = 4$.

In the following sections we consider in turn the restrictions imposed by the positivity condition (1.6), subject to (1.7), for states formed by the action of scalar, vector current and energy-momentum tensor operators on the vacuum. The norms of these states are

expressed in terms of the Euclidean two-point functions for these operators and the matrix elements involving the energy momentum tensor may also be found in terms of the corresponding three-point functions containing the energy momentum tensor as well. These Euclidean correlation functions have been found explicitly previously up to a small number of parameters by making use of conformal invariance [6,7]. The lengthy algebra involved in some of the three-point function computations is reduced significantly by restricting the state so that the three points in the correlation function lie on a straight line. The positivity inequalities obtained in this fashion are checked against the results of direct calculations for free scalar and fermion field theories, which realise conformal invariance for general dimension d, and also for vector field theories for $d = 4$. Applications of the general results constraining the three-point function for the energy momentum tensor are discussed further in the conclusion. We show how our positivity conditions derived for flat space extend to constrain the coefficients of the two independent terms appearing in the trace anomaly for the expectation value of the energy momentum tensor for a conformal field theory in four dimensions on a general curved space background. One has been known to satisfy a positivity condition since it determines the overall scale of the energy momentum tensor two-point function, while the other, which is the coefficient of the topological term in the trace, is known to be positive for free field theory but previously there has been no general argument requiring this.

We should stress that our alternative weak energy condition is motivated by seeing what might be feasible in terms of not being manifestly incorrect in quantum field theory. Even if it is valid it is not at all clear how it might be applied to semi-classical general relativity, although as remarked above it does have implications concerning the energy momentum tensor trace on curved space. Our detailed discussion is restricted to conformally invariant quantum field theories, which includes the case of free massless fields. While we have not undertaken detailed calculations we believe thsat there should be no intrinsic difficulty in extending the analysis to massive free fields.

As mentioned above we illustrate in appendix A the basic philosophy by applying the inequalities to two simple quantum mechanics problems based on a perturbed harmonic oscillator. The positivity condition is verified if the perturbation is not too large and we are thus able to demonstrate that there is no inherent incompatibility between our version of the weak energy condition and the general formalism of quantum mechanics. Appendix B is devoted to proving that no further information is obtained when our calculations

are carried out for more general states than those which lead to a collinear coordinate configurations. This makes essential use of conformal invariance which allows any three points to be transformed to lie on a line.

2. Correlators involving scalar operators

Let us start by recalling some basic results on the constraints imposed by unitarity on two-point correlation functions involving scalar operators, which are equivalent to those following from reflection positivity in Euclidean space.

Initially we consider states formed by the action of a scalar field $\mathcal{O}$, of dimension η, on the vacuum and take

$$|\psi_{\mathcal{O}}\rangle = e^{-H\tau}\mathcal{O}(0,\mathbf{x})|0\rangle\,, \quad \tau>0\,, \tag{2.1}$$

where H is the Hamiltonian given as usual in terms of the energy momentum tensor by

$$H=\int \mathrm{d}^{d-1}x\; T_{00}(0,\mathbf{x})\,. \tag{2.2}$$

In the conformal limit when $g^{\mu\nu}T_{\mu\nu}=0$ it is evident that automatically $\langle 0|T_{\mu\nu}|\psi_{\mathcal{O}}\rangle = 0$. On continuation $x^0 \to -i\tau$ to Euclidean space the two-point function for the scalar field $\mathcal{O}$ is given in the conformal limit by the correlation function

$$\langle \mathcal{O}^E(x)\,\mathcal{O}^E(0)\rangle = C_{\mathcal{O}}\,\frac{1}{x^{2\eta}}\,, \tag{2.3}$$

where x^2 is defined by using the standard Euclidean metric. The norm of the state $|\psi\rangle$ defined by (2.1) is given directly in terms of the Euclidean two-point function (2.3),

$$\langle\psi_{\mathcal{O}}|\psi_{\mathcal{O}}\rangle = \langle \mathcal{O}^E(\tau,\mathbf{x})\,\mathcal{O}^E(-\tau,\mathbf{x})\rangle = C_{\mathcal{O}}\,\frac{1}{(2\tau)^{2\eta}} \geq 0\,, \tag{2.4}$$

which therefore requires positivity of $C_{\mathcal{O}}$ as a consequence of unitarity.

We now turn to analyze the restrictions which follow from eqs. (1.6) and (1.7). As a preliminary result, we need the computation of the Euclidean three-point function involving the energy momentum tensor, which is also uniquely determined by conformal invariance [6]

$$\begin{aligned}&\langle \mathcal{O}^E(x)\,\mathcal{O}^E(y)\,T^E{}_{\alpha\beta}(z)\rangle\\ &\quad = -\frac{C_{\mathcal{O}}}{S_d}\,\frac{d\eta}{d-1}\,\frac{1}{\big((x-z)^2(y-z)^2\big)^{\frac{1}{2}d}\big((x-y)^2\big)^{\eta-\frac{1}{2}d}}\Big(\frac{Z_\alpha Z_\beta}{Z^2}-\frac{1}{d}\delta_{\alpha\beta}\Big)\,,\end{aligned} \tag{2.5}$$

where $S_d = 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$ and

$$Z_\alpha = \frac{(x-z)_\alpha}{(x-z)^2} - \frac{(y-z)_\alpha}{(y-z)^2} \,. \tag{2.6}$$

The overall coefficient in (2.5) is determined in terms of $C_\mathcal{O}$ by Ward identities which relate the three-point function (2.5) to the two-point function given by (2.3). The relation between amplitudes involving tensor operators such as $T_{\mu\nu}$ and the associated Euclidean correlation functions is given by a matrix[3], $\theta_{\mu\alpha}$, so that

$$\langle \psi_\mathcal{O} | T_{\mu\nu}(0) | \psi_\mathcal{O} \rangle = \theta_{\mu\alpha}\theta_{\nu\beta} \langle \mathcal{O}^E(\tau, \mathbf{x})\, \mathcal{O}^E(-\tau, \mathbf{x}) T^E{}_{\alpha\beta}(0, \mathbf{0}) \rangle \,, \tag{2.7}$$

where

$$\theta_{\mu\alpha} = \begin{pmatrix} i & 0 \\ 0 & \delta_{ij} \end{pmatrix} \,. \tag{2.8}$$

Hence from (2.5) it is straightforward to see that

$$\langle \psi_\mathcal{O} | n^\mu n^\nu T_{\mu\nu}(0) | \psi_\mathcal{O} \rangle = \eta \, \frac{C_\mathcal{O}}{S_d} \, \frac{d-1+\mathbf{n}^2}{d-1} \, \frac{1}{(\tau^2 + \mathbf{x}^2)^d (2\tau)^{2\eta - d}} \,. \tag{2.9}$$

Since we assume $\eta > 0$ the positivity condition (1.6) gives nothing new in this case. Moreover, setting $\mathbf{n} = \mathbf{0}$ and using (2.2) gives

$$\langle \psi_\mathcal{O} | H | \psi_\mathcal{O} \rangle = \int \mathrm{d}^{d-1}x \, \langle \psi_\mathcal{O} | T_{00}(0) | \psi_\mathcal{O} \rangle = 2\eta \, \frac{C_\mathcal{O}}{(2\tau)^{2\eta+1}} = -\tfrac{1}{2} \frac{\partial}{\partial \tau} \langle \psi_\mathcal{O} | \psi_\mathcal{O} \rangle \,, \tag{2.10}$$

which provides an additional check on the overall normalisation in (2.5).

3. Correlators involving vector currents

A less trivial example is provided by considering the state formed by the action of a conserved vector operator, which must have dimension $d-1$, on the vacuum. The relation of the matrix elements to Euclidean correlation functions involves applying also the matrix θ, defined in (2.8), to the vector indices and the algebra becomes more complicated. On the other hand, the two-point correlation function for the vector current is still characterised by a single overall parameter in conformal field theories and there are no anomalous dimensions present in this case.

[3] Indices $\alpha, \beta, \gamma, \ldots$ for Euclidean correlation functions are solely subscripts while Minkowski vector indices $\mu, \nu, \ldots$ are raised and lowered with the standard metric $g_{\mu\nu}$, where $g_{00} = -1$, $g_{ij} = \delta_{ij}$.

As in the previous section, we first construct the state

$$|\psi_V\rangle = e^{-H\tau} V_\mu(0,\mathbf{x})|0\rangle\psi^\mu\,, \quad \tau > 0\,. \tag{3.1}$$

We have contracted the V_μ operator with an external vector ψ^μ, which can be chosen freely and which allows positivity conditions to be investigated for all possible combinations for Lorentz indices with ease. The norm of the state is directly related to the Euclidean two-point function

$$\langle\psi_V|\psi_V\rangle = \psi^\mu\psi^\nu\theta_{\mu\alpha}\theta_{\nu\beta}\langle V^E{}_\alpha(\tau,\mathbf{x})\, V^E{}_\beta(-\tau,\mathbf{x})\rangle\,, \tag{3.2}$$

and in the conformal limit we have the simple form

$$\langle V^E{}_\alpha(x)\, V^E{}_\beta(0)\rangle = C_V\,\frac{1}{x^{2(d-1)}} I_{\alpha\beta}(x)\,, \qquad I_{\alpha\beta}(x) \equiv \delta_{\alpha\beta} - 2\frac{x_\alpha x_\beta}{x^2}\,, \tag{3.3}$$

where $I_{\alpha\beta}(x)$ represents the action of inversions. From (3.3) it is easy to see that

$$\langle\psi_V|\psi_V\rangle = C_V\,\frac{1}{(2\tau)^{2(d-1)}}\psi^\mu\psi^\mu\,, \tag{3.4}$$

which is manifestly positive so long as $C_V > 0$. We should note here that the initially surprising sum over two upper μ indices, so that there is an effective Euclidean metric, is essential to ensure positivity.

We now consider the conditions flowing from the assumption of the positivity condition,

$$\langle\psi_V|n^\mu n^\nu T_{\mu\nu}(0)|\psi_V\rangle \geq 0\,, \tag{3.5}$$

since $\langle 0|T_{\mu\nu}(0)|\psi_V\rangle = 0$, without restriction on ψ^μ. The matrix element in (3.5) is directly related to the Euclidean three-point function $\langle V^E{}_\gamma(x)\, V^E{}_\delta(y)\, T^E{}_{\alpha\beta}(z)\rangle$ which in the conformal limit has two possible linearly independent forms [6]. As mentioned in the introduction for simplicity we consider the case when x, y, z are collinear, along the direction defined by $\hat{e}_\alpha = (1,\mathbf{0})$, but appendix B shows that the results obtained for the collinear configuration are equivalent to the more general case assuming $\mathbf{x} \neq \mathbf{0}$. With this simplification the three-point function is restricted to the simple form,

$$\langle V^E{}_\gamma(\hat{x}\hat{e})\, V^E{}_\delta(\hat{y}\hat{e})\, T^E{}_{\alpha\beta}(\hat{z}\hat{e})\rangle = \frac{1}{|\hat{x}-\hat{z}|^d\,|\hat{y}-\hat{z}|^d\,|\hat{x}-\hat{y}|^{d-2}}\,\mathcal{A}_{\gamma\delta\alpha\beta}\,, \tag{3.6}$$

where $\mathcal{A}_{\gamma\delta\alpha\beta} = \mathcal{A}_{\delta\gamma\alpha\beta} = \mathcal{A}_{\gamma\delta\beta\alpha}$, $\mathcal{A}_{\gamma\delta\alpha\alpha} = 0$ is an invariant tensor under $O(d-1)$ transformations leaving $\hat{e}$ invariant. Using the notation $\hat{e}_\gamma \hat{e}_\delta \mathcal{A}_{\gamma\delta\alpha\beta} = \mathcal{A}_{\hat{e}\hat{e}\alpha\beta}$ we may therefore write

$$\mathcal{A}_{\hat{e}\hat{e}mn} = \beta\,\delta_{mn}\,, \quad \mathcal{A}_{i\hat{e}m\hat{e}} = \delta\,\delta_{im}\,, \quad \mathcal{A}_{ijmn} = \rho\,\delta_{ij}\delta_{mn} + \tau(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\,, \tag{3.7}$$

with other components zero or determined by the traceless condition on $\alpha\beta$. The conservation conditions for the vector current and the energy momentum tensor lead to two conditions

$$\beta + \rho + d\tau = 0\,, \qquad 2\delta + d\rho + (d+2)\tau = 0\,, \tag{3.8}$$

so there are left two independent parameters which may be taken as ρ, τ. Ward identities give a relation to the coefficient of the two-point function

$$C_V = \frac{S_d}{d}\big(d\rho + (d+2)\tau\big)\,, \tag{3.9}$$

so this combination must be positive.

If in (3.1) we set $\mathbf{x} = \mathbf{0}$ then using (3.6) we may obtain in the conformal limit

$$\langle \psi_V | n^\mu n^\nu T_{\mu\nu}(0) | \psi_V \rangle = \frac{1}{\tau^{2d}(2\tau)^{d-2}} \psi^\sigma \psi^\rho M_{\sigma\rho}\,, \quad M_{\sigma\rho} = n^\mu n^\nu \theta_{\mu\alpha}\theta_{\nu\beta}\theta_{\sigma\gamma}\theta_{\rho\delta}\mathcal{A}_{\gamma\delta\alpha\beta}\,, \tag{3.10}$$

where explicitly

$$M_{\sigma\rho} = \begin{pmatrix} -(d-1+\mathbf{n}^2)\beta & -2\delta\, n_j \\ -2\delta\, n_i & \big((d-1+\mathbf{n}^2)\rho + 2\tau\big)\delta_{ij} + 2\tau\, n_i n_j \end{pmatrix}. \tag{3.11}$$

The positivity condition (3.5) then reduces to the positivity of the matrix M. For $\mathbf{n} = \mathbf{0}$ this requires

$$-\beta = \rho + d\tau \geq 0\,, \quad (d-1)\rho + 2\tau \geq 0\,, \tag{3.12}$$

and it is easy to see that together they give $C_V > 0$ using (3.9). For $\mathbf{n}$ non zero it is convenient to write $\psi^\sigma = (a, b\hat{\mathbf{n}} + \mathbf{v})$ with $\mathbf{v}\cdot\mathbf{n} = 0$, $\hat{\mathbf{n}} = \mathbf{n}/|\mathbf{n}|$ so that

$$\psi^\sigma \psi^\rho M_{\sigma\rho} = \big((d-1+\mathbf{n}^2)\rho + 2\tau\big)\mathbf{v}^2 + \mathcal{V}^T \mathcal{M} \mathcal{V}\,, \tag{3.13}$$

where

$$\mathcal{V} = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \mathcal{M} = \begin{pmatrix} -(d-1+\mathbf{n}^2)\beta & -2\delta\,|\mathbf{n}| \\ -2\delta\,|\mathbf{n}| & (d-1+\mathbf{n}^2)\rho + 2\tau(1+\mathbf{n}^2) \end{pmatrix}. \tag{3.14}$$

If $\mathbf{n}^2 = 1$ then

$$\det \mathcal{M} = (d-2)\big(d^2\rho + (3d+2)\tau\big)\tau \, . \tag{3.15}$$

The positivity requirements then reduce to $(d-1+\mathbf{n}^2)\rho + 2\tau \geq 0$ and positivity of the eigenvalues of the matrix $\mathcal{M}$, which of course implies $\det \mathcal{M} \geq 0$. It is not difficult to show that the necessary and sufficient conditions, for $d > 2$, can be reduced to

$$d\rho + 2\tau \geq 0 \, , \qquad \tau \geq 0 \, , \tag{3.16}$$

which are stronger than (3.12). It is evident that in this case the positivity conditions on the energy momentum tensor lead to requirements which go beyond what can be found by consideration of the two-point function alone.

It is of course crucial to check whether these conditions are compatible with known results which means comparing with explicit calculations for free fields. The two relevant cases are free n-component scalar fields, in which there are conserved currents corresponding to the $SO(n)$ symmetry, and also Dirac fermions. In both cases for massless fields there is a conserved traceless energy momentum tensor for general d. Neglecting inessential positive constants previous calculations give

$$\rho_S = d-4 \, , \quad \tau_S = d \, ; \qquad\qquad \rho_F = 1 \, , \quad \tau_F = 0 \, . \tag{3.17}$$

Thus the positivity conditions (3.16) or (3.12) are met in both these cases.

The above example is indicative of the general type of results which may arise from energy conditions in quantum field theory. A local energy density positivity condition should, if valid, enforce some constraints on three-point functions involving the energy momentum tensor which in turn transform into inequalities for the parameters defining the three point function at a conformally invariant fixed point. These constraints may be checked against known results, which in practice are restricted to free field theories.

4. Correlators involving energy momentum tensors

We finally consider the case where all correlation functions are made out of energy momentum tensors. This instance is of particular interest since some important parameters characterising the theory show up through properties of this operator, which was

the motivation for this investigation[4]. Furthermore, there are well-established relations between trace anomaly and operator product expansion coefficients, which are computable *via* two and three-point stress tensor correlation functions in four dimensions. Our positivity results are potentially relevant in both of these areas.

We therefore consider the state

$$|\psi_T\rangle = e^{-H\tau} T_{\mu\nu}(0,\mathbf{x})|0\rangle\, \psi^{\mu\nu}\,, \quad \tau > 0\,. \tag{4.1}$$

The two-point function of the energy momentum tensor after analytic continuation to Euclidean space in the conformal limit is given by a simple generalisation of (3.3)

$$\langle T^E{}_{\alpha\beta}(x)\, T^E{}_{\gamma\delta}(0)\rangle = C_T\, \frac{1}{x^{2d}} \Big(\tfrac{1}{2}\big(I_{\alpha\gamma}(x) I_{\beta\delta}(x) + I_{\beta\gamma}(x) I_{\alpha\delta}(x)\big) - \frac{1}{d}\delta_{\alpha\beta}\delta_{\gamma\delta}\Big)\,. \tag{4.2}$$

With this result

$$\langle\psi_T|\psi_T\rangle = \psi^{\sigma\rho}\psi^{\kappa\lambda}\theta_{\sigma\gamma}\theta_{\rho\delta}\theta_{\kappa\epsilon}\theta_{\lambda\eta}\langle T^E{}_{\gamma\delta}(\tau,\mathbf{x})\, T^E{}_{\epsilon\eta}(-\tau,\mathbf{x})\rangle = C_T\, \frac{1}{(2\tau)^{2d}}\psi^{\sigma\rho}\psi^{\sigma\rho}\,, \tag{4.3}$$

if we impose $g_{\mu\nu}\psi^{\mu\nu} = 0$ or $\psi^{00} = \psi^{ii}$. Just as with previous examples, unitarity requires the positivity of C_T.

In order to analyze the more subtle condition (1.6), in conjunction with (1.7), we now set $\mathbf{x} = \mathbf{0}$ in the definition of the state $|\psi_T\rangle$ so that it is easy to see that

$$\langle 0|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle = C_T\, \frac{1}{\tau^{2d}}\, n^\mu n^\nu \psi^{\mu\nu}\,. \tag{4.4}$$

With $\mathbf{x} = \mathbf{0}$ the analysis of the matrix element $\langle\psi_T|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle$ can be reduced to the collinear Euclidean three-point function which can be written simply as

$$\langle T^E{}_{\alpha\beta}(\hat{x}\hat{e})\, T^E{}_{\gamma\delta}(\hat{y}\hat{e})\, T^E{}_{\epsilon\eta}(\hat{z}\hat{e})\rangle = \frac{1}{|\hat{x}-\hat{y}|^d\, |\hat{x}-\hat{z}|^d\, |\hat{y}-\hat{z}|^d}\, \mathcal{A}_{\alpha\beta\gamma\delta\epsilon\eta}\,, \tag{4.5}$$

where $\mathcal{A}_{\alpha\beta\gamma\delta\epsilon\eta}$ is an invariant tensor under $O(d-1)$ rotations preserving $\hat{e}$ symmetric and is traceless for each pair of indices $\alpha\beta$, $\gamma\delta$, $\epsilon\eta$ and also symmetric under interchange of each pair. The essential components, with a similar notation to (3.7), are

$$\begin{aligned}
\mathcal{A}_{ijk\hat{e}m\hat{e}} &= \rho\, \delta_{ij}\delta_{km} + \tau(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\,, \\
\mathcal{A}_{ijk\ell mn} &= r\, \delta_{ij}\delta_{k\ell}\delta_{mn} \\
&\qquad + s\big(\delta_{ij}(\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m}) + \delta_{kl}(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}) + \delta_{mn}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\big) \\
&\qquad + t\big(\delta_{ik}\delta_{jm}\delta_{\ell n} + i \leftrightarrow j, k \leftrightarrow \ell, m \leftrightarrow n\big)\,,
\end{aligned} \tag{4.6}$$

[4] This was first considered some time ago by Cappelli and Latorre [8] and reported on in ref. [9]

with others determined by the symmetry and traceless conditions. The conservation equations give two conditions

$$2\tau + dr + (d+4)s + 2t = 0\,, \quad 2\rho - dr + (d-2)s + 2(d+4)t = 0\,, \tag{4.7}$$

so there remain three independent parameters which may be taken as r, s, t. One linear combination is determined in terms of the coefficient of the two-point function in (4.2) by a Ward identity

$$\begin{aligned} C_T &= \frac{2S_d}{d(d-1)(d+2)}\Big(d\beta - (d-2)(d+1)\epsilon + 2(d-1)\gamma\Big) \\ &= \frac{4S_d}{d(d+2)}\Big(dr + (d^2+2d+4)s + (d^2+5d+2)t\Big)\,, \end{aligned} \tag{4.8}$$

where we have introduced the alternative variables β, ϵ, γ for later convenience by

$$\beta = (d-1)^2 r + 6(d-1)s + 8t\,, \quad -\epsilon = (d-1)s + 4t\,, \quad \gamma = -(d-1)\rho - 2\tau\,. \tag{4.9}$$

With these results we may obtain the analogous expression to (3.10) in this case

$$\langle\psi_T|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle = \frac{1}{\tau^{2d}(2\tau)^d}\,\psi^{\sigma\rho}\psi^{\kappa\lambda}M_{\sigma\rho,\kappa\lambda}\,, \tag{4.10}$$

where $M_{\sigma\rho,\kappa\lambda} = M_{\kappa\lambda,\sigma\rho}$, which forms a $\frac{1}{2}(d+2)(d-1)\times\frac{1}{2}(d+2)(d-1)$ matrix, can be expressed in terms of $\mathcal{A}_{\mu\nu\sigma\rho\kappa\lambda}$ giving

$$\begin{aligned} M_{0m,0n} &= \gamma\,\delta_{mn} - \rho\mathbf{n}^2\delta_{mn} - 2\tau\, n_m n_n\,, \\ M_{0m,k\ell} &= -2\big(\rho\,\delta_{k\ell}n_m + \tau(\delta_{mk}n_\ell + \delta_{m\ell}n_k)\big)\,, \\ M_{ij,k\ell} &= \frac{\beta}{d-1}\delta_{ij}\delta_{k\ell} - \epsilon\Big(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \frac{2}{d-1}\delta_{ij}\delta_{k\ell}\Big) \\ &\quad + r\,\delta_{ij}\delta_{k\ell}\mathbf{n}^2 + s(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})\mathbf{n}^2 + 2s(\delta_{ij}n_k n_\ell + \delta_{k\ell}n_i n_j) \\ &\quad + 2t(n_i n_k \delta_{j\ell} + i\leftrightarrow j, k\leftrightarrow \ell)\,, \end{aligned} \tag{4.11}$$

with other components determined by symmetry and conditions such as $M_{ij,00} = M_{ij,kk}$, giving for instance $M_{00,00} = (d-1+\mathbf{n}^2)\beta$.

Our positivity conditions then reduce to

$$\psi^{\sigma\rho}\psi^{\kappa\lambda}M_{\sigma\rho,\kappa\lambda} \geq 0 \quad \text{if} \quad n^\mu n^\nu\psi^{\mu\nu} = 0\,. \tag{4.12}$$

We analyse this condition in two stages. First we consider the case where $\mathbf{n} = \mathbf{0}$. It is easy to see that (4.12) and tracelessness of ψ yield $\psi^{00} = \psi^{ii} = 0$. Moreover, a short computation shows that our proposed positivity conditions reduce to

$$\gamma \geq 0\,, \quad -\epsilon \geq 0\,, \tag{4.13}$$

although positivity of the full matrix $M_{\sigma\rho,\kappa\lambda}$ would lead to $\beta \geq 0$ as well.

More generally, we consider the case $\mathbf{n} \neq \mathbf{0}$. To disentangle the complete set of independent positivity restrictions, we decompose $\psi^{\sigma\rho}$ in the form

$$\begin{gathered}\psi^{\sigma\rho} = \begin{pmatrix} f + (d-1)g & \frac{1}{2}w_n + \frac{1}{2}a\hat{v}_n + \frac{1}{2}e\hat{n}_n \\ \frac{1}{2}w_m + \frac{1}{2}a\hat{v}_m + \frac{1}{2}e\hat{n}_m & u_{ij} + \frac{1}{2}b(\hat{n}_i\hat{v}_j + \hat{n}_j\hat{v}_i) + f\hat{n}_i\hat{n}_j + g\delta_{ij} \end{pmatrix}, \\ \mathbf{n}\cdot\mathbf{w} = \mathbf{w}\cdot\hat{\mathbf{v}} = \mathbf{n}\cdot\hat{\mathbf{v}} = 0\,, \quad u_{ij} = u_{ji}\,,\ u_{ii} = 0\,,\ u_{ij}\hat{n}_j = 0\,,\end{gathered} \tag{4.14}$$

so that

$$\psi^{\sigma\rho}\psi^{\sigma\rho} = 2(f + \tfrac{1}{2}dg)^2 + \tfrac{1}{2}d(d-2)g^2 + \tfrac{1}{2}(e^2 + a^2 + b^2) + \tfrac{1}{2}\mathbf{w}^2 + u_{ij}u_{ij}\,, \tag{4.15}$$

and

$$n^\sigma n^\rho \psi^{\sigma\rho} = (1 + \mathbf{n}^2)f + (d - 1 + \mathbf{n}^2)g + |\mathbf{n}|e\,. \tag{4.16}$$

Using (4.14) we may find

$$\psi^{\sigma\rho}\psi^{\alpha\beta}M_{\sigma\rho,\alpha\beta} = 2(-\epsilon + s\mathbf{n}^2)u_{ij}u_{ij} + (\gamma - \rho\mathbf{n}^2)\mathbf{w}^2 + \mathcal{V}_2{}^T\mathcal{M}_2\mathcal{V}_2 + \mathcal{V}_3{}^T\mathcal{M}_3\mathcal{V}_3\,, \tag{4.17}$$

where

$$\mathcal{V}_2 = \begin{pmatrix} a \\ b \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} \gamma - \rho\mathbf{n}^2 & -2\tau|\mathbf{n}| \\ -2\tau|\mathbf{n}| & -\epsilon + (s + 2t)\mathbf{n}^2 \end{pmatrix}. \tag{4.18}$$

and, assuming

$$\mathcal{V}_3 = \begin{pmatrix} e \\ f \\ g \end{pmatrix}, \tag{4.19}$$

then

$$\mathcal{M}_3 = \begin{pmatrix} \gamma - (\rho + 2\tau)\mathbf{n}^2 & -2(d\rho + 4\tau)|\mathbf{n}| & 2d\gamma|\mathbf{n}| \\ -2(d\rho + 4\tau)|\mathbf{n}| & X & \frac{d^2}{d-1}(d - 1 + \mathbf{n}^2)\beta - 2d\frac{d-2}{d-1}\epsilon\mathbf{n}^2 \\ 2d\gamma|\mathbf{n}| & \frac{d^2}{d-1}(d - 1 + \mathbf{n}^2)\beta - 2d\frac{d-2}{d-1}\epsilon\mathbf{n}^2 & d^2(d - 1 + \mathbf{n}^2)\beta \end{pmatrix} \tag{4.20}$$

where

$$X = \frac{1}{d-1}\big(d^2\beta - 2(d-2)\epsilon\big) + \frac{1}{d-1}\big((d+1)\beta - 4(d-2)\epsilon\big)\mathbf{n}^2 + (r+6s+8t)\mathbf{n}^2 \, .$$

The positivity condition (4.12) as applied to (4.17) requires positivity of $-\epsilon + s\mathbf{n}^2$, $\gamma - \rho\mathbf{n}^2$ as well as of the matrices $\mathcal{M}_2$ and ${\mathcal{V}_3}^T\mathcal{M}_3\mathcal{V}_3$, where the vector $\mathcal{V}_3$ is constrained by the condition $n^\sigma n^\rho \psi^{\sigma\rho} = 0$, for all $|\mathbf{n}|$ such that $0 \le |\mathbf{n}| \le 1$. In particular, positivity of $\mathcal{M}_2$ for $\mathbf{n} = \mathbf{0}$ leads to our previous result (4.13). Setting $|\mathbf{n}| = 1$ the positivity conditions from $\psi^{\sigma\rho}\psi^{\alpha\beta}M_{\sigma\rho,\alpha\beta} \ge 0$, excluding those involving the matrix $\mathcal{M}_3$, reduce to the linear relations

$$-\epsilon + s = ds + 4t \ge 0 \, , \qquad \gamma - \rho = -d\rho - 2\tau \ge 0 \, , \tag{4.21}$$

as well as the nonlinear condition

$$Q \equiv \det \mathcal{M}_2\big|_{|\mathbf{n}|=1} = \det \begin{pmatrix} -d\rho - 2\tau & -2\tau \\ -2\tau & ds + 6t \end{pmatrix} \ge 0 \, . \tag{4.22}$$

Only the conditions arising from $\mathcal{M}_3$ are sensitive to the requirement $n^\sigma n^\rho \psi^{\sigma\rho} = 0$ since from (4.16) this constrains only $\mathcal{V}_3$. For $|\mathbf{n}| = 0$ then we may eliminate f in terms of g, $f = -(d-1)g$, so that the previous results (4.13) are recovered. If we take $|\mathbf{n}| = 1$ then we choose to eliminate e in favour of f and g and the matrix acting on $\begin{pmatrix} f \\ g \end{pmatrix}$ becomes

$$\overline{\mathcal{M}} = (d-2) \begin{pmatrix} (d+4)(dr+4s-4t) & d(d+3)(dr+4s-4t) \\ d(d+3)(dr+4s-4t) & \frac{1}{2}d^2\big(d(2d+3)r + 3(3d+4)s - 6(d+2)t\big) \end{pmatrix} . \tag{4.23}$$

If $d > 2$ the positivity of $\overline{\mathcal{M}}$ reduces to

$$I \equiv dr + 4s - 4t = \frac{d^2\beta - 4(d-1)\gamma - 2(d-2)\epsilon}{(d-1)(d-2)(d+3)} \ge 0 \, , \tag{4.24}$$

and also, by considering $\det \overline{\mathcal{M}}$,

$$J \equiv -d(d+6)r + (d^2 - 24)s + 2(d^2 + 6d + 12)t \ge 0 \, . \tag{4.25}$$

Note that, from (4.8),

$$C_T = \frac{2S_d}{d^2(d+2)}\Big((d-2)(d+3)I + (d+2)\big(2\gamma - (d-2)\epsilon\big)\Big) \, , \tag{4.26}$$

which by virtue of (4.13) and (4.24) is manifestly positive as is necessary for unitarity.

The detailed analysis of the mutual interdependence of the constraints obtained in this case is obviously more complicated than previously. It is perhaps of interest first to consider the case $d = 3$ separately since, as shown in [6], there are then only two independent parameters which may be taken as

$$u = r - 4t \,, \qquad v = s + 2t \,. \tag{4.27}$$

Note that $C_T = \frac{16}{15}\pi(3u + 19v)$ and for the quantities appearing in the various inequalities (if $d = 3$ then u_{ij} in (4.14) is absent so that there is no condition for $-\epsilon + s$),

$$-\epsilon = \tfrac{1}{8}\gamma = v \,, \quad I = 3u + 4v \,, \quad J = -3(9u + 5v) \,, \quad \gamma - \rho = \tfrac{1}{2}(17v - 3u) \,. \tag{4.28}$$

The inequalities lead to $-\frac{5}{9} \geq u/v \geq -\frac{4}{3}$.

When $d = 4$, which is of primary interest, three parameters are necessary but the positivity conditions are homogeneous so that they essentially constrain their ratios. The relationship between different conditions is most easily visualised in terms of the two-dimensional plot given by Fig. 1.

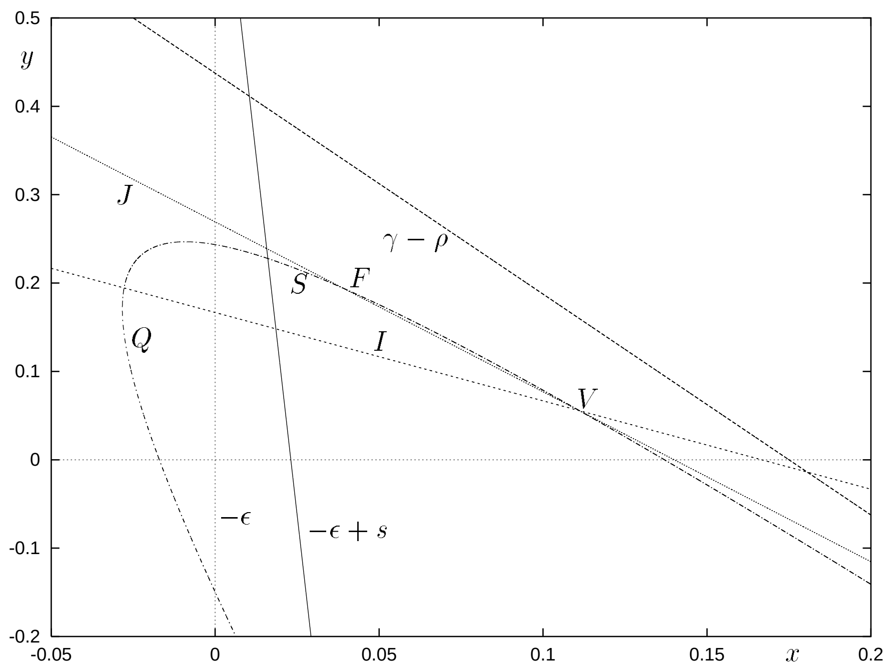

Fig. 1 Boundaries of various positivity inequalities for $d = 4$ on graph with coordinates $y = \beta/(\beta - 5\epsilon + 3\gamma)$ and $x = -\epsilon/(\beta - 5\epsilon + 3\gamma)$, where $\beta - 5\epsilon + 3\gamma \propto C_T > 0$. Each boundary is labelled by the corresponding quantity which is required to be positive on the side for which it is positive. The allowed region satisfied by all constraints is the central triangular region with one corner cut off by the $Q = 0$ curve. The points labelled by V, F, which lie on the boundary of the positivity region, and S correspond to free vector, fermion and scalar fields.

The relevance and consistency of our results may be checked by investigating the cases of the essentially trivial conformal field theories formed by free scalars, fermions, for general d, and also vector fields when $d = 4$. Neglecting inessential positive constants we have from [6]

$$\begin{aligned} r_S &= d^3 + 28d - 16\,, \quad s_S = d(d^2 - 8d + 4)\,, \quad t_S = d^3\,, \\ r_F &= -4\,, \qquad s_F = d\,, \qquad t_F = 0\,, \\ d &= 4\,, \qquad r_V = -3\,, \qquad s_V = 2\,, \qquad t_V = -1\,. \end{aligned} \tag{4.29}$$

We have used these results to check all the various inequalities, for $d = 4$ they give the points shown in Fig. 1. For general free conformal field theories for $-\epsilon + s$, which featured in (4.21), we find

$$(-\epsilon + s)_S = d^2(d-2)^2\,, \qquad (-\epsilon + s)_F = d^2\,, \qquad (-\epsilon + s)_V = 4\,, \tag{4.30}$$

so this inequality is respected in each case and also with the definition in (4.22)

$$Q_S = d^4(d+2)(d-1)(d-2)^3\,, \quad Q_F = \tfrac{1}{2}d^4(d+2)\,, \quad Q_V = 0\,. \tag{4.31}$$

For the matrix $\mathcal{M}_3$ (4.20) for $|\mathbf{n}| = 1$ we find

$$\begin{aligned} \det \mathcal{M}_{3,S} &= 2d^6(d+2)(d-1)(d-2)^4(d^5 + d^4 - 10d^3 - 4d^2 - 24d + 32)\,, \\ \det \mathcal{M}_{3,F} &= \det \mathcal{M}_{3,V} = 0\,. \end{aligned} \tag{4.32}$$

It is easy to see that $\det \mathcal{M}_{3,S} < 0$ when $d = 3$. Nevertheless from the expressions for I, J in the more restricted conditions (4.24) and (4.29) we find

$$\begin{aligned} I_S &= d^2(d+2)(d-2)\,, \qquad I_F = I_V = 0\,, \\ J_S &= 2d^2(d-2)(d^2 + d - 10)\,, \quad J_F = d^2(d+4)\,, \quad J_V = 0\,, \end{aligned} \tag{4.33}$$

which are positive for $d = 3$. This result shows how the restriction on the states $|\psi\rangle$ to the subspace defined by (1.7) is sufficient to maintain the fulfilment of the positivity conditions in free scalar theories in three dimensions. In terms of the variables in (4.27) we have $(u/v)_S = -\frac{13}{21}$, $(u/v)_F = -\frac{4}{3}$ which lie in the required positivity range.

However J_S in (4.33) is no longer positive for d close to two although when $d = 2$ exactly this problem is absent since there is just one unique expression for the conformally invariant energy momentum tensor three-point function whose coefficient is determined by the value of C_T, which is proportional to the Virasoro central charge c. Nevertheless the lack of positivity of J_S if $2 < d < \frac{1}{2}(\sqrt{41} - 1)$ suggests, following the guidelines exemplified by the quantum mechanical analogues discussed in Appendix A, that further restrictions are necessary.

5. A refined energy density positivity postulate

The problems noted above for scalar field theories are confined to the matrix $\overline{\mathcal{M}}$ defined in (4.23). As mentioned above it is natural to introduce additional constraints on the state $|\psi_T\rangle$. Any further such conditions will require some knowledge of the detailed dynamics and will therefore be model dependent to this extent. As a possible extra constraint on the states $|\psi\rangle$, which may be natural in the context of conformal field theories, we postulate, as well as (1.7),

$$\langle\psi_{\mathcal{O}}|n^\mu n^\nu T_{\mu\nu}(0)|\psi\rangle = 0\,, \tag{5.1}$$

where $|\psi_{\mathcal{O}}\rangle$ is constructed as in (2.1) in terms of a scalar field $\mathcal{O}$ appearing in the theory. It is useful to note that

$$\langle \mathcal{O}^E(x)\mathcal{O}'^E(y){T^E}_{\alpha\beta}(z)\rangle = 0 \ \text{ if } \ \eta \neq \eta'\,, \quad \langle \mathcal{O}^E(x){V^E}_\gamma(y){T^E}_{\alpha\beta}(z)\rangle = 0\,, \tag{5.2}$$

and hence this extra condition will not affect the positivity condition (3.5) for states formed by a vector operator which led to (3.12) or (3.16)but will impose only restrictions on any conditions arising from the matrix $\overline{\mathcal{M}}$ as required.

To apply the condition (5.1) to the states $|\psi_T\rangle$ we make use of results for the Euclidean three-point function $\langle \mathcal{O}^E(x){T^E}_{\gamma\delta}(y){T^E}_{\alpha\beta}(z)\rangle$ which in the conformal limit has only one linearly independent form. In the collinear configuration we have

$$\langle \mathcal{O}^E(\hat{x}\hat{e}){T^E}_{\gamma\delta}(\hat{y}\hat{e}){T^E}_{\alpha\beta}(\hat{z}\hat{e})\rangle = \frac{1}{|\hat{x}-\hat{y}|^\eta|\hat{x}-\hat{z}|^\eta|\hat{y}-\hat{z}|^{2d-\eta}}\,\mathcal{B}_{\gamma\delta\alpha\delta}\,, \tag{5.3}$$

where $\mathcal{B}_{\gamma\delta\alpha\delta}$ is an $O(d-1)$ invariant tensor given by (note that here δ, ϵ, γ are new parameters unrelated to those of the previous section)

$$\mathcal{B}_{ijk\ell} = \delta\,\delta_{ij}\delta_{k\ell} + \epsilon(\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk})\,, \quad \mathcal{B}_{i\hat{e}k\hat{e}} = \gamma\,\delta_{ik}\,, \tag{5.4}$$

with other components determined by symmetry and tracelessness. Conservation of the energy momentum tensor gives two conditions

$$2\gamma = -(d-\eta)\big((d-1)\delta+2\epsilon\big)\,, \qquad (d-\eta)\gamma = -d\,\delta-(d+2)\epsilon\,, \tag{5.5}$$

which may be solved, up to an inessential overall constant, by taking

$$\epsilon = \tfrac{1}{2}(d-\eta)^2 - \frac{d}{d-1}\,, \quad \delta = \frac{1}{d-1}\big(-(d-\eta)^2+d+2\big)\,, \quad \gamma = -\tfrac{1}{2}(d-\eta)\frac{(d-2)(d+1)}{d-1}\,. \tag{5.6}$$

We can now write, if $\mathbf{x}=\mathbf{0}$ so that using (5.3) is valid,[5]

$$\langle\psi_{\mathcal{O}}|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle = \frac{1}{\tau^{2d}(2\tau)^\eta}\, n^\mu n^\nu \psi^{\sigma\rho}\theta_{\mu\gamma}\theta_{\nu\delta}\theta_{\sigma\alpha}\theta_{\rho\beta}\mathcal{B}_{\gamma\delta\alpha\delta}\,. \tag{5.7}$$

Using (5.4) we can write, with $g_{\mu\nu}\psi^{\mu\nu}=0$,

$$\begin{aligned} n^\mu n^\nu \psi^{\sigma\rho}\theta_{\mu\gamma}\theta_{\nu\delta}\theta_{\sigma\alpha}\theta_{\rho\beta}\mathcal{B}_{\gamma\delta\alpha\delta} \qquad\qquad & \\ = \delta\, d(d-1+\mathbf{n}^2)\psi^{ii} + 2\epsilon\big((d+\mathbf{n}^2)\psi^{ii} + n^i n^j\psi^{ij}\big) - 4\gamma\,\psi^{0i}n^i & \\ = \eta\,\frac{d-2}{d-1}\Big(\big((\eta-d)\mathbf{n}^2 + d+1+\mathbf{n}^2\big)f + (d+1)(d-1+\mathbf{n}^2)g\Big)\,, & \end{aligned} \tag{5.8}$$

where we have used (4.14) and (5.6) as well as $n^\sigma n^\rho\psi^{\sigma\rho}=0$ to eliminate e if $\mathbf{n}\neq\mathbf{0}$. If $\mathbf{n}=\mathbf{0}$ then this vanishes if $\psi^{ii}=0$ which is the same condition as that already obtained from $\langle 0|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle=0$.

In order to analyse the consequences of the extra constraint (5.1) for $|\psi\rangle\to|\psi_T\rangle$ we set $|\mathbf{n}|=1$ and then, instead of positivity of the matrix $\overline{\mathcal{M}}$, we require now only that

$$\mathcal{V}_\eta{}^T\,\overline{\mathcal{M}}\,\mathcal{V}_\eta \geq 0\,, \qquad \mathcal{V}_\eta = \begin{pmatrix} -d(d+1) \\ \eta+2 \end{pmatrix}. \tag{5.9}$$

If η were a free parameter this would reduce to positivity of $\overline{\mathcal{M}}$ again but in any conformal theory the spectrum of scale dimensions η is bounded below by some positive number. For free scalars, which produced the earlier difficulties, the lowest dimension scalar operator which could contribute here is ϕ^2 having dimension $d-2$. Inserting the results (4.29) we find

$$\overline{\mathcal{M}}_S = (d-2)^2 d^2 \begin{pmatrix} (d+2)(d+4) & d(d+2)(d+3) \\ d(d+2)(d+3) & d^2(d^2+5d+2) \end{pmatrix} \tag{5.10}$$

so that

$$\mathcal{V}_{d-2}\,\overline{\mathcal{M}}_S\,\mathcal{V}_{d-2} = (d-2)^2 d^4(d^3+d^2+10d+8)\,, \tag{5.11}$$

which is always positive. For free fermions or, if $d=4$, free vectors

$$\overline{\mathcal{M}}_F = \begin{pmatrix} 0 & 0 \\ 0 & \tfrac12 d^4(d-2) \end{pmatrix}, \qquad \overline{\mathcal{M}}_V = 0\,, \tag{5.12}$$

which satisfy the previous positivity conditions without further restriction. In these theories the lowest dimension scalar operator which may be relevant has a dimension of at least $\eta=d$.

[5] In general the matrix element may be written as $\langle\psi_{\mathcal{O}}(\mathbf{x}',\tau')|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T(\mathbf{x},\tau)\rangle$ but only if $\mathbf{x}'=\mathbf{x},\tau'=\tau$ can conformal invariance be used to transform this to the collinear configuration, as in appendix B, without effectively transforming n^μ. It is perhaps simpler to consider just the state $|\mathcal{O}\rangle=\lim_{\tau\to\infty}(2\tau)^\eta e^{-H\tau}\mathcal{O}(0,\mathbf{x})|0\rangle$, which has norm from (2.4) $\langle\mathcal{O}|\mathcal{O}\rangle=C_{\mathcal{O}}$, and impose $\langle\mathcal{O}|n^\mu n^\nu T_{\mu\nu}(0)|\psi\rangle=0$ instead of (5.1) as an additional condition on the states $|\psi\rangle$ to which the positivity condition is applied. In this situation we can restrict our attention to the collinear case.

6. Constraints on the trace anomaly coefficients

At this point we may summarise our main results and try to exploit them in some related issues that go beyond flat space correlation functions. In curved space-times negative energy densities appear to play an essential role in such effects as Hawking radiation and are also connected with lack of stability of time independent vacuum states. Conversely, some requirement of positive energy density might be desired in order to ensure a stable vacuum and to avoid seeming pathologies like causality violations associated with wormholes. We have, thus, tried to resurrect energy conditions in quantum field theory by postulating a restricted version of the weak energy condition, here initially for flat space-time as a first step towards finding possible generalisations of classical conditions. To circumvent the simplest known counterexamples of violation of such kinds of positivity conditions we have looked for inspiration in two quantum mechanical models. This leads us to consider the linear conditions, defining a restricted subspace,

$$\langle\psi|T|\psi\rangle \geq 0 \quad \text{where} \quad \langle\phi_i|n^\mu n^\nu T_{\mu\nu}|\psi\rangle = 0\,, \tag{6.1}$$

for some suitable natural set of states $|\phi_i\rangle$, on which positivity may then be imposed. It is of course essential that the subspace be defined in a not too model dependent fashion, although even if the number of necessary conditions becomes infinite this does not necessarily lead to them having no content. The constraints imposed by a simple version of this postulate which we have investigated in bosonic, fermionic and vector free field theories, for a restricted set of states $|\psi\rangle$, have not led to any apparent violation but also have given conditions with a non trivial content.

An important corollary of the above results, and one of our main motivations, is the generation of constraints on the coefficients of the trace anomaly in curved space-time backgrounds. As is well known the expectation value of the trace of the energy momentum tensor is then non zero even if it vanishes as an operator on flat space-time, as required in conformal field theories. For $d = 4$ the trace anomaly is formed from dimension 4 scalars constructed from the Riemann tensor and has the general form,

$$g^{\mu\nu}\langle T_{\mu\nu}\rangle_g = -\beta_a F - \beta_b G + h\nabla^2 R\,, \tag{6.2}$$

where $F = C^{\alpha\beta\gamma\delta}C_{\alpha\beta\gamma\delta}$ is the square of the Weyl tensor, $G = \frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}R^{\alpha\beta}{}_{\mu\nu}R^{\gamma\delta}{}_{\sigma\rho}$ is the Euler density and the third term is a total derivative. The coefficient h is in fact

arbitrary since it can be altered by adding a local term to the effective action so we concentrate only on β_a and β_b.

The relation between the coefficient of the Weyl term and the overall coefficient of the two-point function for the energy momentum tensor is well known [10],

$$-\beta_a = \frac{\pi^2}{640} C_T = \frac{\pi^4}{5760}(2\beta - 5\epsilon + 3\gamma) \geq 0\,, \tag{6.3}$$

using the result (4.8) for $d = 4$. This term in the trace anomaly is therefore related to the c-number term appearing in the operator product expansion for two energy momentum tensors and may be regarded as measuring the reaction of the field theory to a shear transformation. The positivity of $-\beta_a$ is thus required by the positivity of C_T which is rooted in the basic unitarity properties of the Hilbert space, as illustrated directly by (4.3).

The trace anomaly constrains the form of the expectation value of the energy momentum tensor on curved space-times, although it does not essentially determine it [11,4] unlike the case in two dimensions. For solutions of Einstein equations $F = G$ so the anomaly depends only on $\beta_a + \beta_b$ which has opposite signs for different free field theories and hence there are no obvious positivity constraints for the anomaly coefficients, such as might be found by considering the flux of Hawking radiation, to be obtained by analysis of $\langle T_{\mu\nu}\rangle_g$ in applications to general relativity in four dimensions.

On the other hand, the coefficient β_b may play a significant role in flat space quantum field theories. In many respects it is the direct analogue of the Virasoro central charge c for two dimensional theories. Its analysis is more involved and requires more sophistication than β_a since its footprint in a flat space quantum field theory lies in three-point correlators of the energy momentum tensor and, consequently, its presumptive positivity can only be derived in terms of some positivity restriction on the energy momentum tensor such as we have postulated. On conformally flat spaces $F = 0$ and then $\int \mathrm{d}^4x\sqrt{g}g^{\mu\nu}\langle T_{\mu\nu}\rangle_g$ is proportional to β_b, the integral of the Euler density is a topological invariant. Cardy [12] suggested that β_b might be a candidate for proving a c-theorem in four dimensions. A significant step towards this aim would be to demonstrate its positivity, at least at renormalisation group fixed points of interacting quantum field theories.

The crucial question is therefore whether the results in section 4 and 5 can lead to the positivity of β_b. Using the work of [6,7], we can relate this coefficient to the parameters in

energy momentum tensor three-point function as follows

$$\begin{aligned}\beta_b &= \frac{\pi^4}{5760}(4r+48s+53t) \\ &= \frac{\pi^4}{5760}\frac{1}{6}(2\beta-95\epsilon-3\gamma)\end{aligned} . \tag{6.4}$$

In order to discuss the issue of positivity of β_b in our framework there are several alternatives. From the conditions for $\mathbf{n}=\mathbf{0}$ which are given in (4.13), supplemented by the positivity of C_T which is given in (4.8), we cannot deduce $\beta_b>0$. The constraints in (4.21), (4.24) and (4.29) are stronger. When $d=4$ we have

$$-\epsilon+s=\tfrac{2}{105}(-2\beta-115\epsilon-9\gamma)\,,\quad \gamma-\rho=\tfrac{1}{15}(-2\beta+5\epsilon+21\gamma)\,,\quad I=\tfrac{2}{21}(4\beta-\epsilon-3\gamma)\,, \tag{6.5}$$

and since

$$2\beta-95\epsilon-3\gamma=\tfrac{10}{63}(4\beta-\epsilon-3\gamma)+\tfrac{16}{21}(-2\beta-115\epsilon-9\gamma)+\tfrac{13}{9}(2\beta-5\epsilon+3\gamma)\,, \tag{6.6}$$

we evidently have from (6.4) that a variety of combinations of inequalities imply $\beta_b>0$ as a corollary of the version of the weak energy condition postulated in this paper. In terms of Fig. 1, $\beta_b>0$ corresponds to $y>\frac{1}{4}(1-100x)$ which is easily seen to be satisfied by the allowed positivity region.

The fact that the conditions leading to this result fail at the peculiar value of d discussed in section 4 for scalar field theories is perhaps indicative that further restrictions are necessary. If we drop all conditions arising from the matrix $\mathcal{M}_3$ and impose just that C_T, which is given by (4.8), is positive as an additional condition we cannot obtain $\beta_b>0$. One more inequality which must involve $\mathcal{M}_3$ directly is necessary. From section 6 we may consider applying the extra condition (5.1) for some η which still leads to some extra conditions. As an illustration for a free scalar theory we may take $\eta=d-2$ and get the constraint

$$8\beta+5\epsilon+\gamma\geq 0\,. \tag{6.7}$$

This supplementary equation is now sufficient to find a manifestly positive expression for β_b, *e.g.* for instance

$$2\beta-95\epsilon-3\gamma=\tfrac{3}{8}(2\beta-5\epsilon+3\gamma)+\tfrac{5}{6}(-2\beta-115\epsilon-9\gamma)+\tfrac{17}{120}(-2\beta+5\epsilon+21\gamma)+\tfrac{2}{5}(8\beta+5\epsilon+\gamma)\geq 0\,. \tag{6.8}$$

In conclusion it is perhaps worth noting that our proposed energy conditions, while speculative to an extent, are susceptible of further tests. In some instances the inequalities such as (3.12), (3.16) or (4.13), (4.21), (4.22) and (4.24) are such that free field theories lie on the boundary. In such cases it is possible that perturbative calculations for interacting theories will verify whether the inequalities are still satisfied. Nevertheless it is perhaps remarkable that, as exhibited in Fig. 1, that the various inequalities severely constrain the parameters of the general energy momentum tensor three point function in a consistent fashion which is compatible with the results of free field theory and in a fashion which implies $\beta_b > 0$.

Acknowledgements

We are grateful to the British Council and Spanish MEC for financial support allowing for travel to our respective institutions, also CICYT under contract AEN95-0590, CIRIT under contract GRQ93-1047. We are also very pleased to acknowledge valuable discussions with Rolf Tarrach.

Appendix A.

As an illustration that our positivity condition makes sense in at least some circumstances we consider two elementary standard quantum mechanical examples.

First we consider the following operator

$$T = \cosh 2\phi \, a^\dagger a + \tfrac{1}{2} \sinh 2\phi \, (a^2 + a^{\dagger 2}) \,, \tag{A.1}$$

where $a, a^\dagger$ are conventional annihilation, creation operators. The expression for T is obviously modelled on the expected form for free field theory. As is well known T can be diagonalised by a Bogoliubov transformation

$$T = a'^\dagger a' - \sinh^2 \phi \,, \tag{A.2}$$

where

$$a' = U a U^{-1} = \cosh \phi \, a + \sinh \phi \, a^\dagger \,, \tag{A.3}$$

with U a unitary operator given by

$$U = e^{\frac{1}{2}\phi(a^2 - a^{\dagger 2})} = (\cosh \phi)^{-\frac{1}{2}} e^{-\frac{1}{2}\tanh\phi \, a^{\dagger 2}} e^{\rho a^\dagger a} e^{\frac{1}{2}\tanh\phi \, a^2} \,, \quad \rho = -\ln \cosh \phi \,. \tag{A.4}$$

For the standard Fock space vacuum $|0\rangle$, satisfying $a|0\rangle = 0$, it is evident that the conditions in (1.1) are met if $\phi \neq 0$. For a general state $|\psi\rangle$ in the Fock space

$$\langle \psi | T | \psi \rangle = \sum_{n=0} {a_n}^2 \big(n - \sinh^2 \phi \big) \quad \text{for} \quad |\psi\rangle = \sum_{n=0} a_n U |n\rangle \,. \tag{A.5}$$

Clearly we can choose the state $|\psi\rangle$ such that $\langle\psi|T|\psi\rangle < 0$ without difficulty since the $n = 0$ term in the sum is always negative. It is also possible to see from (A.4) that

$$\begin{aligned} \langle 0 | T | \psi \rangle &= \sum_{n=0} a_n \big(n - \sinh^2 \phi \big) \langle 0 | U | n \rangle \\ &= (\cosh\phi)^{-\frac{1}{2}} \sum_{n=0} a_{2n} \, \frac{\sqrt{(2n)!}}{2^n n!} \, \big(2n - \sinh^2 \phi \big) \tanh^n \phi \,. \end{aligned} \tag{A.6}$$

Imposing the condition (1.5) does not constrain a_n for odd n so that it is necessary to impose the condition

$$\sinh^2 \phi \leq 1 \,, \tag{A.7}$$

if the positivity condition (1.2) subject to (1.5) is to be at all satisfied. The condition (1.5) may then be regarded as an equation determining a_0 giving

$$a_0 \sinh^2\phi = \sum_{n=1} a_{2n} \, \frac{\sqrt{(2n)!}}{2^n n!} \big(2n - \sinh^2\phi\big) \tanh^n\phi \, . \tag{A.8}$$

Using this to eliminate a_0 in (A.5) then after some algebra we find

$$\begin{aligned}\langle\psi|T|\psi\rangle = {} & \sum_{n=0} a_{2n+1}^2 \big(2n+1-\sinh^2\phi\big) \\ & + \sum_{n=1} \frac{2n+1}{2n} \big(2n - \sinh^2\phi\big)\tanh^2\phi \\ & \times \bigg(a_{2n} - \frac{1}{2n+1} \, \frac{2^n n!}{\sqrt{(2n)!}} \sum_{m=n+1} a_{2m} \, \frac{\sqrt{(2m)!}}{2^m m!} \, \frac{2m - \sinh^2\phi}{\sinh^2\phi} \tanh^{m-n}\phi \bigg)^2 ,\end{aligned} \tag{A.9}$$

which is obviously positive subject to (A.7).

An even simpler second illustration is given by considering the operator

$$T = a^\dagger a + \lambda(a + a^\dagger) = a'^\dagger a' - \lambda^2 \, , \tag{A.10}$$

where

$$a' = a + \lambda = U a U^{-1} \, , \quad U = e^{\lambda(a - a^\dagger)} = e^{-\frac{1}{2}\lambda^2} e^{-\lambda a^\dagger} e^{\lambda a} \, . \tag{A.11}$$

As above a general state can be represented as $|\psi\rangle = \sum_{n=0} a_n U|n\rangle$ and then

$$\langle\psi|T|\psi\rangle = \sum_{n=0} {a_n}^2 (n - \lambda^2) \, . \tag{A.12}$$

In this case imposing the restriction $\langle 0|T|\psi\rangle = 0$ gives

$$a_0 \lambda^2 = \sum_{n=1} a_n \frac{1}{\sqrt{n!}} \, (n - \lambda^2) \lambda^n \, , \tag{A.13}$$

and thus eliminating a_0 from (A.12) gives

$$\langle\psi|T|\psi\rangle = \sum_{n=1} \frac{n-\lambda^2}{n} \, \lambda^2 \bigg(a_n - \sqrt{n!} \sum_{m=n+1} a_m \frac{1}{\sqrt{m!}} \, (m - \lambda^2) \, \lambda^{m-n-2} \bigg)^2 . \tag{A.14}$$

In consequence we have the positivity condition (1.2), subject to (1.5), if $\lambda^2 < 1$.

These examples demonstrate that the positivity conditions suggested here may be valid for an operator T which is close to an operator T_0 which annihilates $|0\rangle$. As the difference between T and T_0 becomes larger more and more conditions on the state $|\psi\rangle$ are necessary if $\langle\psi|T|\psi\rangle \geq 0$ is to be maintained.

Appendix B.

In this appendix we show that conformal invariance ensures that there is no loss of generality in deriving positivity conditions for the states $|\psi_V\rangle$ and $|\psi_T\rangle$, defined in (3.1) and (4.1), by setting $\mathbf{x} = \mathbf{0}$. This is essentially because a conformal transformation on Euclidean space allows any three points to be made collinear so that we are able then to use the simple expressions for the three-point functions given by (3.6) and (4.5).

To demonstrate this we first note that under a conformal transformation on Euclidean space for which $x \to x'$, where $\mathrm{d}x'_\alpha \mathrm{d}x'_\alpha = \Omega(x)^{-2}\mathrm{d}x_\alpha \mathrm{d}x_\alpha$, a Euclidean vector field transforms as $V \to V'$ where $V'_\alpha(x') = \Omega(x)^{d-1}\mathcal{R}(x)_{\alpha\beta}V_\beta(x)$ and we define the orthogonal matrix $\mathcal{R}$ by $\partial x'_\alpha/\partial x_\beta = \Omega(x)^{-1}\mathcal{R}_{\alpha\beta}(x)$. For a special conformal transformation

$$x'_\alpha = \frac{x_\alpha + b_\alpha x^2}{\Omega(x)}\,, \quad \Omega(x) = 1 + 2b{\cdot}x + b^2x^2\,. \tag{B.1}$$

If $b_\alpha = (0,\mathbf{b})$ then the plane $x_\alpha \hat{e}_\alpha = 0$ is left invariant and we may choose $\mathbf{b} = \mathbf{x}/(\tau^2+\mathbf{x}^2)$ so as to transform $(\hat{x},\mathbf{0}) \to (\tau,\mathbf{x})$. Assuming conformal invariance then from (3.6) we may obtain

$$\langle V^E{}_\gamma(\tau,\mathbf{x})\, V^E{}_\delta(-\tau,\mathbf{x})\, T^E{}_{\alpha\beta}(0)\rangle = \mathcal{R}_{\gamma\gamma'}(\tau,\mathbf{x})\mathcal{R}_{\delta\delta'}(-\tau,\mathbf{x})\, \frac{1}{(\tau^2+\mathbf{x}^2)^d(2\tau)^{d-2}}\, \mathcal{A}_{\gamma'\delta'\alpha\beta}\,, \tag{B.2}$$

where

$$\mathcal{R}_{\alpha\beta}(\tau,\mathbf{x}) = \begin{pmatrix} \frac{\tau^2-\mathbf{x}^2}{\tau^2+\mathbf{x}^2} & -\frac{2\tau x_j}{\tau^2+\mathbf{x}^2} \\ \frac{2\tau x_i}{\tau^2+\mathbf{x}^2} & \delta_{ij} - \frac{2x_ix_j}{\tau^2+\mathbf{x}^2} \end{pmatrix}\,. \tag{B.3}$$

By taking into account the action of the matrix θ defined by (2.8) we now find instead of (3.10)

$$\langle\psi_V|n^\mu n^\nu T_{\mu\nu}(0)|\psi_V\rangle = \frac{1}{(\tau^2+\mathbf{x}^2)^d(2\tau)^{d-2}}\,\psi^\sigma\psi^\rho\,\tilde{\mathcal{R}}_\sigma{}^{\sigma'}(\tau,\mathbf{x})^*\tilde{\mathcal{R}}_\rho{}^{\rho'}(\tau,\mathbf{x})M_{\sigma'\rho'}\,, \tag{B.4}$$

with now the matrix $\tilde{\mathcal{R}}$ given by

$$\tilde{\mathcal{R}}_\mu{}^\nu(\tau,\mathbf{x}) = \begin{pmatrix} \frac{\tau^2-\mathbf{x}^2}{\tau^2+\mathbf{x}^2} & \frac{2i\tau x^j}{\tau^2+\mathbf{x}^2} \\ \frac{2i\tau x_i}{\tau^2+\mathbf{x}^2} & \delta_i^j - \frac{2x_i x^j}{\tau^2+\mathbf{x}^2} \end{pmatrix} , \tag{B.5}$$

which satisfies $\tilde{\mathcal{R}}_\mu{}^\sigma(\tau,\mathbf{x})\tilde{\mathcal{R}}_\mu{}^\rho(\tau,\mathbf{x})g_{\sigma\rho} = g_{\mu\nu}$. It is evident that positivity of (B.5) also requires $M_{\sigma\rho}$ to be a positive matrix just as (3.10). It is also of interest to consider the integral over $\mathbf{x}$ which gives

$$\begin{aligned} &\int \mathrm{d}^{d-1}x \, \langle\psi_V|n^\mu n^\nu T_{\mu\nu}(0)|\psi_V\rangle \\ &\qquad = 2S_d \frac{1}{(2\tau)^{2d-1}} \frac{1}{d}\Big\{(\psi^0)^2\big(M_{00}+M_{ii}\big) + 2\psi^0\psi^i\big(M_{0i}+M_{i0}\big) + \psi^i\psi^i M_{00} \\ &\qquad\qquad + \frac{1}{d+1}\big(\psi^i\psi^i M_{jj} + d(d-1)\psi^i\psi^j M_{ij}\big)\Big\} . \end{aligned} \tag{B.6}$$

If we set $\mathbf{n}=\mathbf{0}$, so that $M_{0i}=M_{i0}=0,\ M_{ij}\propto\delta_{ij}$, then it is easy to see that from (3.4)

$$\langle\psi_V|H|\psi_V\rangle = 2S_d\frac{1}{(2\tau)^{2d-1}}\frac{1}{d}\,\psi^\mu\psi^\mu M_{\sigma\sigma} = -\tfrac{1}{2}\frac{\partial}{\partial\tau}\langle\psi_V|\psi_V\rangle , \tag{B.7}$$

if, using (3.4),

$$(d-1)C_V = \frac{S_d}{d}\,M_{\sigma\sigma} . \tag{B.8}$$

This is of course in agreement with the Ward identity result (3.9). It is perhaps worth noting that the unitarity condition $C_V>0$ constrains the trace of the matrix $M_{\sigma\rho}$ while positivity of the matrix element of T_{00} gives positivity of the whole matrix when $\mathbf{n}=\mathbf{0}$. An additional constraint on the matrix may be obtained by considering the momentum operator

$$P_i = -\int \mathrm{d}^{d-1}x\, T_{0i}(0,\mathbf{x}) . \tag{B.9}$$

By expanding (B.6) to first order in $\mathbf{n}$, and using (3.11), we may find

$$\langle\psi_V|\mathbf{P}|\psi_V\rangle = 2S_d\frac{1}{(2\tau)^{2d-1}}\frac{4}{d}\,\delta\,\psi^0\boldsymbol{\psi} = -4C_V\frac{1}{(2\tau)^{2d-1}}\,\psi^0\boldsymbol{\psi} , \tag{B.10}$$

where the second expression results by using $\mathbf{P}$ as a generator of translations in $\mathbf{x}$. Clearly (B.10) requires $2S_d\delta = -dC_V$ in agreement with (3.8),(3.9).

In a similar fashion for the state $|\psi_T\rangle$ defined in (4.1) for arbitrary $\mathbf{x}$ we find

$$\begin{aligned} &\langle\psi_T|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle \\ &= \frac{1}{(\tau^2+\mathbf{x}^2)^d(2\tau)^d}\,\psi^{\sigma\rho}\psi^{\kappa\lambda}\,\tilde{\mathcal{R}}_\sigma{}^{\sigma'}(\tau,\mathbf{x})^*\tilde{\mathcal{R}}_\rho{}^{\rho'}(\tau,\mathbf{x})^*\tilde{\mathcal{R}}_\kappa{}^{\kappa'}(\tau,\mathbf{x})\tilde{\mathcal{R}}_\lambda{}^{\lambda'}(\tau,\mathbf{x})M_{\sigma'\rho',\kappa'\lambda'} , \end{aligned} \tag{B.11}$$

and furthermore, instead of (4.4), we now have

$$\langle 0|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle = C_T\,\frac{1}{(\tau^2+\mathbf{x}^2)^d}\,n^\mu n^\nu \tilde{\mathcal{R}}_\mu{}^\sigma(\tau,\mathbf{x})\tilde{\mathcal{R}}_\nu{}^\rho(\tau,\mathbf{x})\psi^{\sigma\rho}\,. \tag{B.12}$$

Clearly we gain no more from positivity conditions for $\mathbf{x}\neq\mathbf{0}$ than we have found from (4.12). From (B.12) we then have

$$\int \mathrm{d}^{d-1}x\,\langle 0|n^\mu n^\nu T_{\mu\nu}(0)|\psi_T\rangle = C_T\,\frac{2S_d}{d+1}\big((d-1)n^i n^j\psi^{ij} - \mathbf{n}^2\psi^{ii}\big)\,, \tag{B.13}$$

which is in accord with the requirement that $\langle 0|P^\mu|\psi_T\rangle = 0$ where $P^\mu = (H,\mathbf{P})$.

References

[1] S.W. Hawking and G.F.R. Ellis, The Large Scale Structure of Space-time, Cambridge University Press, Cambridge 1973, chap. 4.

[2] H. Epstein, V. Glaser and A. Jaffe, Nuovo Cimento **36** (1966) 1016.

[3] P.C.W. Davies and S.A. Fulling, Proc. Roy. Soc. **A356** (1977) 237.

[4] M. Visser, Phys. Rev. **D54** (1996) 5103, 5116, 5123; gr-qc/9604007,9604008, 9604009 and gr-qc/9703001.

[5] L.H. Ford and T.A. Roman, Phys. Rev. **D46** (1992) 1328, **D51** (1995) 4277, **D53** (1996) 1988, **D55** (1997) 2082; gr-qc/9607003;
M.J. Pfenning and L.H. Ford, Phys. Rev. **D55** (1997) 4813; gr-qc/9608005;
É.É. Flanaghan, gr-qc/9706006.

[6] H. Osborn and A. Petkou, Ann. Phys. **231** (1994) 311; hep-th/9307010.

[7] J. Erdmenger and H. Osborn, Nucl. Phys. **B483** (1997) 431; hep-th/9605009.

[8] A. Cappelli and J.I. Latorre, unpublished.

[9] H. Osborn, *in* Proceedings of the XXVII Ahrenshoop International Symposium, DESY 94-053.

[10] A. Cappelli, D. Friedan and J.I. Latorre, Nucl. Phys. **B352** (1991) 616;
A. Cappelli, J.I. Latorre and X. Vilasis-Cardona, Nucl. Phys. **B376** (1992) 510; hep-th/9109041.

[11] S.M. Christensen and S.A. Fulling, Phys. Rev. **D15** (1977) 2088

[12] J.L. Cardy, Phys. Lett. **B215** (1988) 749.